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Element-Based Node Selection Method for Reduction of Eigenvalue Problems

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Dynamic analysis of structural systems requires a considerable amount of computing time. Because the dynamic behavior of a structure is dominated by lower modes, all of the eigenvalues of the system need not be calculated. The previously proposed methods approximate the eigenvalues of a global system but they take great amount of computing time to construct a reduced system. An element-level energy estimation technique is proposed to construct a reduced finite element model. Through several examples, it is demonstrated that the proposed method effectively saves computational time and accurately predicts the eigenvalues of a global system.

Nomenclature

e	=	relative error of eigenvalue
$[\boldsymbol{K}]$	=	global stiffness matrix
$[\boldsymbol{K}_G]$	=	reduced stiffness matrix of Guyan's[2] reduced system
$[\boldsymbol{K}_{\mathrm{IRS}}]$	=	reduced stiffness matrix of the improved reduced system
$[\boldsymbol{M}]$	=	global mass matrix
$[\boldsymbol{M}_G]$	=	reduced mass matrix of Guyan's[2] reduced system
$[\boldsymbol{M}_{\mathrm{IRS}}]$	=	reduced mass matrix of the improved reduced system
N_e	=	total number of elements
$\boldsymbol{Ra}$	=	Rayleigh quotient of the global system
$\boldsymbol{Ra}_e$	=	Rayleigh quotient of an element level
$[\boldsymbol{T}_{\mathrm{IRS}}]$	=	transformation matrix of improved reduced system
$\boldsymbol{w}_e^i$	=	ith element weighting factor
$\{z^{(k)}\}$	=	normalized kth Ritz vector
λ	=	eigenvalue of the global system
λ_r	=	eigenvalue of the reduced system
Π_e^i	=	modified Rayleigh quotient
$\{\varphi\}$	=	eigenvector of the global system
$\{\varphi_p\}$	=	primary degrees of freedom
$\{\varphi_s\}$	=	slave degrees of freedom
Ψ_i	=	inner production of eigenvector in ith element

I. Introduction

THE eigenvalue problem of large structural systems requires a considerable amount of solving time. Until now, many approximate techniques have been developed to calculate the eigenvalues in reduced form. These schemes approximated the lower eigenvalues that characterize the global behavior of structures. Noor reviewed this field of research.[1] One of the static condensation methods was introduced by Guyan.[2] In this method, the degrees of freedom that do not significantly influence the solution field are eliminated. Leung proposed a method reducing the order of dynamic matrices exactly.[3] Suarez introduced the dynamic condensation method with an iterative scheme.[4] Also, Paz determined the frequencies of interest by another iterative scheme.[5] O'Callahan[6] improved Guyan's method by considering the first-order approximation terms in the transformation formula of the slave degrees of freedom. Though O'Callahan's method provides a better result than that of Guyan, it may have a nonpositive definite mass matrix by the wrong selection of the master degrees of freedom. Zhang and Li proposed the succession-level approximate reduction method.[7] This method presents better results than the previous two methods because the slave transformation formula is improved iteratively. This method has been improved by Kim and Kang, who supplemented the slave transformation formula to accelerate the convergence with the second-order approximation.[8]

Although these techniques can greatly reduce the computational effort by eliminating degrees of freedom, they suffer from nontrivial flaws. Because they ignore several higher-order terms, the results of these methods are highly dependent on the choice of the master degrees of freedom. If the master degrees of freedom are not selected properly, the accuracy of the results would not be guaranteed. For this reason, several techniques have been proposed to select the master degrees of freedom effectively. The most effective technique is the sequential elimination method (SEM). This technique selects the degrees of freedom whose ratio of stiffness to mass in the diagonal terms of mass and stiffness matrices is highest. This method was initially proposed by Henshell and Ong[9] and Ong.[10] In the similar way, Shah and Raymund selected the master degrees of freedom analytically.[11] Matta has also adjusted the remaining degrees of freedom to compensate the effect of each eliminated degree of freedom.[12] Kidder has selected those by the bandwidth approach.[13] In particular, SEM demonstrates effective selection of the master degrees of freedom; the prediction of the lower mode eigenvalues is very reliable. One by one, it eliminates the slave degrees of freedom whose ratio of the stiffness to mass given by $[\boldsymbol{K}_{ii}]/[\boldsymbol{M}_{ii}]$ is the highest. Because it eliminates only a single degree of freedom in each iteration, it requires considerable computing time in selecting the master degrees of freedom. Recently, Kim and Choi[14] proposed a selection method of the master degrees of freedom by each energy row sum of matrix. In the sequential selection, the degrees of freedom with the largest energy are chosen in each mode, and by the element-level energy estimation the degrees of freedom with the largest row sum are selected. This method sometimes improperly selects the master degrees of freedom. If the energies of lower modes have very small values compared to those of higher modes, the degrees of freedom that are selected by the energy estimation in each row show the overemphasis of lower eigenvalues. If there is a great difference in the order of magnitude between the lowest frequency and the next higher frequency, the weighting factor does not significantly effect the selection of master degrees of freedom. If the master degrees of freedom are not selected appropriately, only a few eigenvalues are computed accurately because first only a few lower eigenvalues are overemphasized.

In the present study, a new selection method of the master nodes is proposed by using the energy estimation of each element instead of each node. Ritz vectors of each mode are used to calculate the kinetic and the strain energy in each element. Ritz vectors have the properties of orthogonality and mass normalization. The strain

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and kinetic energies of each element, $u_e^T K_e u_e/2$ and $u_e^T M_e u_e/2$, respectively, are calculated by using the degrees of freedom that are extracted from the Ritz vectors. The Rayleigh quotient of an element-level $\boldsymbol{Ra}_e = u_e^T K_e u_e / u_e^T M_e u_e$ is calculated from the strain and kinetic energies. Elements with smaller quotient $\boldsymbol{Ra}_e$ are selected, and the nodes connected to the selected elements are chosen as master nodes. The slave degrees of freedom in the present study are expressed in terms of master degrees of freedom. The slave degrees are approximated by considering up to the first-order term of eigenvalue. The selection time of the master degrees of freedom and the computing time of the reduced eigenvalue problems are compared with the solving time of the full eigenvalue problem.

Usually, only the lower-order modes in the conventional structural dynamics problems are considered. In the large-scaled problems, the reduction method is effective in eigenvalue analysis. How the master degrees of freedom are selected among all of the active degrees of freedom is the key issue in the reduction method. The proposed method in the present study provides a criterion for the selection of the master nodes in large-scaled structures.

II. Condensation of Eigenvalue Problem

In finite element analysis, for undamped free vibration, a general eigenvalue problem is presented as follows:

$$[\boldsymbol{K}]\varphi = \lambda[\boldsymbol{K}]\varphi \tag{1}$$

Equation (1) can be expressed by master and slave degrees of freedom as

$$\begin{bmatrix} \boldsymbol{K}_{pp} & \boldsymbol{K}_{ps} \\ \boldsymbol{K}_{sp} & \boldsymbol{K}_{ss} \end{bmatrix} \begin{Bmatrix} \varphi_p \\ \varphi_s \end{Bmatrix} = \lambda \begin{bmatrix} \boldsymbol{M}_{pp} & \boldsymbol{M}_{ps} \\ \boldsymbol{M}_{sp} & \boldsymbol{M}_{ss} \end{bmatrix} \begin{Bmatrix} \varphi_p \\ \varphi_s \end{Bmatrix} \tag{2}$$

From Eq. (2), the relation between the master and the slave degrees of freedom can be obtained as

$$\begin{aligned} [\boldsymbol{K}_{pp}]\{\varphi_p\} + [\boldsymbol{K}_{ps}]\{\varphi_s\} &= \lambda[\boldsymbol{M}_{pp}]\{\varphi_p\} + \lambda[\boldsymbol{M}_{ps}]\{\varphi_s\} \\ [\boldsymbol{K}_{sp}]\{\varphi_p\} + [\boldsymbol{K}_{ss}]\{\varphi_s\} &= \lambda[\boldsymbol{M}_{sp}]\{\varphi_p\} + \lambda[\boldsymbol{M}_{ss}]\{\varphi_s\} \end{aligned} \tag{3}$$

From the second equation of Eqs. (3), the slave degrees of freedom can be expressed by the master degrees of freedom:

$$\{\varphi_s\} = -([\boldsymbol{K}_{ss}] - \lambda[\boldsymbol{M}_{ss}])^{-1}([\boldsymbol{K}_{sp}] - \lambda[\boldsymbol{M}_{sp}])\{\varphi_p\} = [\boldsymbol{T}(\lambda)]\{\varphi_p\} \tag{4}$$

Equation (4) contains the matrix inversion, $([\boldsymbol{K}_{ss}] - \lambda[\boldsymbol{M}_{ss}])^{-1}$. This term can be approximated by series expansion as follows:

$$\begin{aligned} ([\boldsymbol{K}_{ss}] - \lambda[\boldsymbol{M}_{ss}])^{-1} = \big[&\boldsymbol{I} + \lambda[\boldsymbol{K}_{ss}]^{-1}[\boldsymbol{M}_{ss}] + \left(\lambda[\boldsymbol{K}_{ss}]^{-1}[\boldsymbol{M}_{ss}]\right)^2 \\ &+ \left(\lambda[\boldsymbol{K}_{ss}]^{-1}[\boldsymbol{M}_{ss}]\right)^3 + \cdots\big][\boldsymbol{K}_{ss}]^{-1} \end{aligned} \tag{5}$$

When the first-order term of λ is included, the slave degrees of freedom can be expressed as

$$\{\varphi_s\} \cong -\left(\boldsymbol{I} + \lambda[\boldsymbol{K}_{ss}]^{-1}[\boldsymbol{M}_{ss}]\right)[\boldsymbol{K}_{ss}]^{-1}([\boldsymbol{K}_{sp}] - \lambda[\boldsymbol{M}_{sp}])\{\varphi_p\} \tag{6}$$

From Eq. (6) when the first-order term is considered, the transformation equation is given as follows:

$$\begin{aligned} \{\varphi_s\} \cong& -[\boldsymbol{K}_{ss}]^{-1}[\boldsymbol{K}_{sp}]\{\varphi_p\} \\ &+ \lambda[\boldsymbol{K}_{ss}]^{-1}\left([\boldsymbol{M}_{sp}] - [\boldsymbol{M}_{ss}][\boldsymbol{K}_{ss}]^{-1}[\boldsymbol{K}_{sp}]\right)\{\varphi_p\} \end{aligned} \tag{7}$$

Because the eigenvalue λ in Eq. (7) mainly related to the lower modes is an unknown value, Guyan's static condensation can be used to replace the eigenvalue λ. In Guyan's static condensation, an eigenvalue problem can be obtained in terms of the master degrees of freedom as follows:

$$[\boldsymbol{K}_G]\{\varphi_p\} = \lambda[\boldsymbol{K}_G]\{\varphi_p\} \tag{8}$$

$$\begin{aligned} [\boldsymbol{K}_G] &= [\boldsymbol{K}_{pp}] - [\boldsymbol{K}_{ps}][\boldsymbol{K}_{ss}]^{-1}[\boldsymbol{K}_{sp}] \\ [\boldsymbol{M}_G] &= [\boldsymbol{M}_{pp}] - [\boldsymbol{M}_{ps}][\boldsymbol{T}_0] + [\boldsymbol{T}_0]^T[\boldsymbol{M}_{sp}] + [\boldsymbol{T}_0]^T[\boldsymbol{M}_{ss}][\boldsymbol{T}_0] \\ [\boldsymbol{T}_0] &= -[\boldsymbol{K}_{ss}]^{-1}[\boldsymbol{K}_{sp}] \end{aligned} \tag{9}$$

In Eq. (8), $\lambda[\varphi_p]$ is expressed as $[\boldsymbol{M}_G]^{-1}[\boldsymbol{K}_G]\{\varphi_p\}$. This relation is substituted into Eq. (7). Finally, the slave degrees of freedom $\{\varphi_s\}$ is expressed in terms of the primary degrees of freedom $\{\varphi_p\}$ as

$$\begin{aligned} \{\varphi_s\} &= -[\boldsymbol{K}_{ss}]^{-1}[\boldsymbol{K}_{sp}]\{\varphi_s\} \\ &\quad + [\boldsymbol{K}_{ss}]^{-1}\left([\boldsymbol{M}_{sp}] - [\boldsymbol{M}_{ss}][\boldsymbol{K}_{ss}]^{-1}[\boldsymbol{K}_{sp}]\right)[\boldsymbol{M}_G]^{-1}[\boldsymbol{K}_G]\{\varphi_p\} \\ &= [\boldsymbol{T}_{\mathrm{IRS}}]\{\varphi_p\} \end{aligned} \tag{10}$$

where

$$\begin{aligned} [\boldsymbol{T}_s] &= -[\boldsymbol{K}_{ss}]^{-1}[\boldsymbol{K}_{sp}] \\ [\boldsymbol{T}_{\mathrm{IRS}}] &= [\boldsymbol{T}_s] + [\boldsymbol{K}_{ss}]^{-1}([\boldsymbol{M}_{sp}] + [\boldsymbol{M}_{ss}][\boldsymbol{T}_s])[\boldsymbol{M}_G]^{-1}[\boldsymbol{K}_G] \end{aligned} \tag{11}$$

The matrices $[\boldsymbol{T}_s]$, $[\boldsymbol{K}_{ss}]^{-1}([\boldsymbol{M}_{sp}] + [\boldsymbol{M}_{ss}][\boldsymbol{T}_s])$, and $[\boldsymbol{M}_G]^{-1}[\boldsymbol{K}_G]$ are obtained using effective direct solvers for a sparse system. The improved reduced system is indicated by the subscript IRS. In the present study, a band solver is used, and it does not take much time for calculation. When the preceding equations are used, the reduced eigenvalue problem can be reconstructed as follows:

$$\{\varphi\} = \begin{Bmatrix} \varphi_p \\ \varphi_s \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{I} \\ \boldsymbol{T}_{\mathrm{IRS}} \end{Bmatrix}\{\varphi_p\} = \{\boldsymbol{T}\}\{\varphi_p\} \tag{12}$$

Equation (12) is substituted into the Rayleigh quotient $\boldsymbol{Ra}$, and it can be given as

$$\boldsymbol{Ra} = \frac{\{\varphi\}^T[\boldsymbol{K}]\{\varphi\}}{\{\varphi\}^T[\boldsymbol{M}]\{\varphi\}} = \frac{\{\varphi_p\}^T[\boldsymbol{T}]^T[\boldsymbol{K}][\boldsymbol{T}]\{\varphi_p\}}{\{\varphi_p\}^T[\boldsymbol{T}]^T[\boldsymbol{M}][\boldsymbol{T}]\{\varphi_p\}} \tag{13}$$

From Eq. (13), the reduced stiffness and mass matrix can be defined as follows:

$$[\boldsymbol{K}_{\mathrm{IRS}}] = [\boldsymbol{T}]^T[\boldsymbol{K}][\boldsymbol{T}], \qquad [\boldsymbol{M}_{\mathrm{IRS}}] = [\boldsymbol{T}]^T[\boldsymbol{M}][\boldsymbol{T}] \tag{14}$$

By the use of Eq. (14), the reduced eigenvalue problem can be constructed and easily solved:

$$[\boldsymbol{K}_{\mathrm{IRS}}]\{\varphi_p\} = \lambda_r[\boldsymbol{M}_{\mathrm{IRS}}]\{\varphi_p\} \tag{15}$$

where $[\boldsymbol{K}_{\mathrm{IRS}}]$ and $[\boldsymbol{M}_{\mathrm{IRS}}]$ are defined as follows:

$$\begin{aligned} [\boldsymbol{K}_{\mathrm{IRS}}] &= [\boldsymbol{K}_{pp}] + [\boldsymbol{T}_{\mathrm{IRS}}]^T[\boldsymbol{K}_{sp}] + [\boldsymbol{K}_{ps}][\boldsymbol{T}_{\mathrm{IRS}}] + [\boldsymbol{T}_{\mathrm{IRS}}]^T[\boldsymbol{K}_{ss}][\boldsymbol{T}_{\mathrm{IRS}}] \\ [\boldsymbol{M}_{\mathrm{IRS}}] &= [\boldsymbol{M}_{pp}] + [\boldsymbol{T}_{\mathrm{IRS}}]^T[\boldsymbol{M}_{sp}] + [\boldsymbol{M}_{ps}][\boldsymbol{T}_{\mathrm{IRS}}] + [\boldsymbol{T}_{\mathrm{IRS}}]^T[\boldsymbol{M}_{ss}][\boldsymbol{T}_{\mathrm{IRS}}] \end{aligned} \tag{16}$$

III. Element-Based Energy Estimation

The Rayleigh quotient is used to estimate energy level of each element. The results of selection without consideration of weighting factors generate the undesirable tendency of overemphasis of the lower eigenvalues. Therefore, we use weighting factors to select the elements considering higher eigenvalues properly, as well as the lower eigenvalues. After selecting the master elements, the reduction of degrees of freedom is carried out at element level.

A. Calculation of Ritz Vectors

To estimate the energy of each element, Ritz vectors need to be constructed first. The initial vector $\{\boldsymbol{x}^{(1)}\}^*$ is approximated as being the diagonal term of the mass matrix[15]:

$$[\boldsymbol{K}]\left\{\boldsymbol{x}^{(1)}\right\}^* = [\boldsymbol{M}_{ii}] \tag{17}$$

From Eq. (17) the first static vector can be obtained, and it is normalized to find the first Ritz vector $\{\boldsymbol{x}^{(1)}\}$. The procedure of normalization is given as follows:

$$\left\{\boldsymbol{x}^{(1)}\right\}^T[\boldsymbol{M}]\left\{\boldsymbol{x}^{(1)}\right\} = 1 \tag{18}$$

First, the Ritz vector is used to find the second static vector. The new static vector becomes M orthogonal to the static vectors obtained by the Gram–Schmidt procedure, which were obtained from the previous steps, and is normalized as Eq. (18). The procedure of orthogonalization is

$$\left\{x^{(i)}\right\} = \left\{x^{(i)}\right\} - \sum_{k=1}^{i-1}\left[\left\{x^{(k)}\right\}^T [M]\left\{z^{(k)}\right\}\right]\left\{z^{(k)}\right\}$$

$$\left\{z^{(k)}\right\} = \frac{\left\{x^{(k)}\right\}}{\sqrt{\left\{x^{(k)}\right\}^T [M]\left\{x^{(k)}\right\}}} \tag{19}$$

The procedure given in Eqs. (17–19) will be repeated until the kth Ritz vector is obtained. When the eigenmode needs to be calculated, the exact eigenmode can be obtained by transformation using the Ritz matrix. For the selection of master elements, the exact eigenmode does not have to be calculated. Thus, the master element selection criterion is based only on the Rayleigh quotient $\boldsymbol{Ra}_e$ of each element.

B. Rayleigh Quotient of Each Element

When the n-Ritz vectors are constructed from the procedure just described, the sum of the Rayleigh quotients of each element i can be calculated as follows:

$$\boldsymbol{Ra}_e^i = \sum_{k=1}^{n} \frac{\left\{z^{(k)}\right\}^T\left[\boldsymbol{K}_e^i\right]\left\{z^{(k)}\right\}}{\left\{z^{(k)}\right\}^T\left[\boldsymbol{M}_e^i\right]\left\{z^{(k)}\right\}} \tag{20}$$

where i is the element number and $[\boldsymbol{K}_e^i]$ and $[\boldsymbol{M}_e^i]$ are element-level matrices. We are supposed to select m master elements. The energy-level estimator based on the Rayleigh quotient $\boldsymbol{Ra}_e^i$ of ith element can be obtained as summing the Rayleigh quotient calculated by each Ritz vector from the first to the nth. After estimating $\boldsymbol{Ra}_e^i$ for all of the elements in the full domain, the elements with the smallest first m Rayleigh quotients $\boldsymbol{Ra}_e^i$ are selected as master elements.

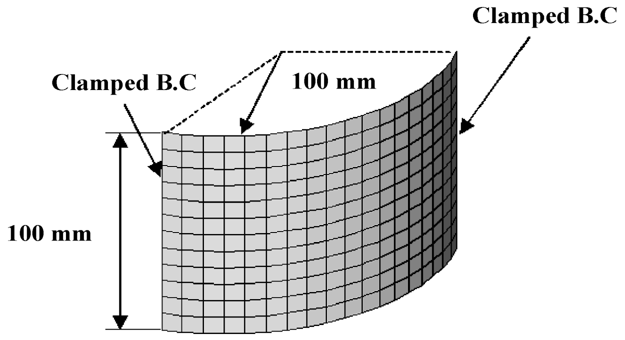


Fig. 1 Configuration and analysis condition of the cylindrical panel: $E = 3 \times 10^5$ Pa, density = 0.01 kg/m^3, Poisson ratio = 0.3, thickness = 1 mm, 216 elements, and 247 nodes.

C. Consideration of Weighting Factor

A weighting factor is needed to avoid the overemphasis on lower eigenvalues. A proper consideration of the weighting factors provides the eigenvalue accurately from the first to the nth mode. The weighting factor $\boldsymbol{w}_e^i$ is obtained through the ratio of diagonal terms, $[\boldsymbol{K}_{ii}]/[\boldsymbol{M}_{ii}]$. If $[\boldsymbol{K}]$ and $[\boldsymbol{M}]$ are the matrices of $n \times n$ elements, the total size of the matrix $[\boldsymbol{K}_{ii}]/[\boldsymbol{M}_{ii}]$ becomes $(n \times 1)$. The degrees of freedom corresponding to each element are extracted from the matrix the entry of which is $[\boldsymbol{K}_{ii}]/[\boldsymbol{M}_{ii}]$, and then the same number of inner production is obtained as the number of elements. The weighting factor is equal to the square of the each value divided by the maximum value of $([\boldsymbol{K}_{ii}]/[\boldsymbol{M}_{ii}])$:

$$\boldsymbol{w}_e^i = (\psi_i/\psi_{\max})^2, \qquad \psi_i = \left(\left[\boldsymbol{K}_{ii}^e\right]/\left[\boldsymbol{M}_{ii}^e\right]\right)^T\left(\left[\boldsymbol{K}_{ii}^e\right]/\left[\boldsymbol{M}_{ii}^e\right]\right) \tag{21}$$

where $i = 1 \sim N_e$. The final modified Rayleigh quotient Π_e^i with weighing factor is expressed as

$$\Pi_e^i = \sum_{l=1}^{k} \boldsymbol{Ra}_e^i \boldsymbol{w}_e^i \tag{22}$$

This novel selection index is proposed to select the master degrees of freedom.

IV. Numerical Examples

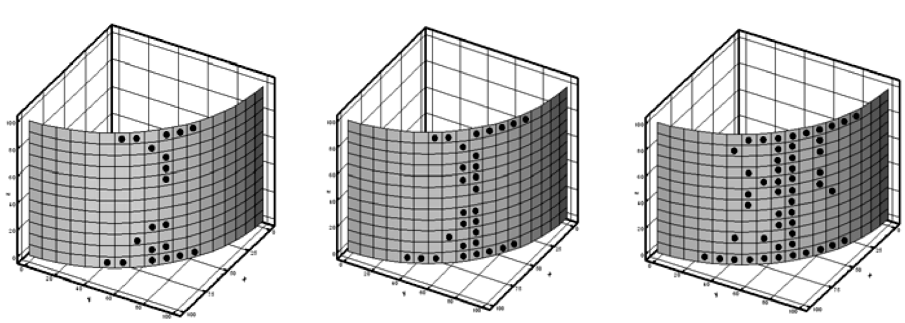
This section presents examples to illustrate the effectiveness of the proposed method to improve the method of condensation. Three numerical examples are considered. The configuration of the problem and the results are given in each example. The results of the condensed eigenvalue problems are compared to those of a full system. The condensed eigenvalue problem is very effective. Required computing time of the present selection method is compared to that of SEM. The relative error for eigenvalue is defined as

$$e(\%) = \left|\left(\lambda_r^i - \lambda^i\right)/\lambda^i\right| \times 100(i : \text{mode number}) \tag{23}$$

where i is the mode number. The master elements selected in the analysis, are indicated by circular black dots.

A. Cylindrical Panel

The first example deals with the cylindrical panel shown in Fig. 1. The cylindrical panel is constrained at the both sides by the clamped boundary condition, and it contains a total of 216 elements. To calculate the Rayleigh quotient of each element, first 120 Ritz vectors are obtained. With use of these Ritz vectors, the selection results of 20, 30, and 50 elements are shown in Fig. 2. Ritz vector extraction procedure takes only 41 s. Figures 2a–2c show the locations of the selected elements. In the first case (Fig. 2a), the selection procedure takes 5.56 s and the solve time for the reduced matrix is 9.54 s. By the use of the inverse power method, 50 eigenvalues are obtained. It takes 196 s to solve the full matrix. The cases shown in Figs. 2a–2c, cases a–c do not require much time for calculating Ritz vectors and for selecting elements. In calculating the eigenvalues of the reduced system, the case show in Fig. 2c just takes a little bit more time than the cases a and b shown in Figs. 2a and 2b.



a) 20 elements **b) 30 elements** **c) 50 elements**

Fig. 2 Selected elements (black dots) of cylindrical panel by ELSM.

Figures 3a–3c show the locations of the selected elements using the result of the full-system eigenvectors. The locations of the selected elements are quite similar to those of Fig. 2. They are mainly distributed in the center area of the system. Although the result of the element selection by Ritz vectors shows a little difference from those given in Figs. 3a–3c, the global selection tendency of the present method is similar to that of the full system. Figure 3d shows the element selection result in the distorted mesh. The eigenvalue result of the distorted mesh configuration is compared with that of the regular mesh, as shown subsequently. Figure 4 shows the relative errors of eigenvalues by changing the number of the selected elements. The reduced systems by selected elements are constructed with IRS. The relative error is given in Eq. (23). In Fig. 4, the relative error of case a increases over 5% after the 25th mode. The relative error above the 30th mode increases rapidly. However, by selecting sufficient number of elements, one can obtain reliable results. For 30 selected elements, the relative error is very small up to 35th modes, at which the error is around 5%; for 50 selected elements, there is only 2∼4% relative error.

Table 1 shows the time comparison according to the selection number of elements. The solving time of reduced systems shows only small differences with each case. However, the reduced system does not require much time in calculating the eigenvalues, unlike a full system. The present method is more effective than SEM for constructing the reduced system. Table 2 shows the constructing time of the reduced system and calculation time by SEM. SEM can provide reliable predictions of eigenvalues through an iterative algorithm. However, SEM takes an excessively large amount of time to select the primary degrees of freedom (DOF). As shown in Table 2, the selection of 444 DOF takes about 1350 s with an iterative algorithm. It has same DOF as 40 selected elements, as shown in Table 1. By the element-level selection method (ELSM), the required time can be reduced to 1/20. Figure 5 shows the nodes associated with primary DOF selected by SEM. Figure 6 shows the comparison of the relative error in the distorted and regular mesh between ELSM and SEM. The eigenvalues of the reduced system are provided by IRS and Guyan's methods. In SEM, Guyan's method and the IRS method assure accuracy in both lower range and higher range modes. However ELSM cannot guarantee the accuracy of the eigenvalues of the reduced system by Guyan's method. The computing time of both methods is of the same order. Thus, it is recommended to use the IRS method in computing eigenvalues in the present ELSM. In this example, ELSM is not severely affected by the mesh size or distortion tendency. In Fig. 6, although SEM can give reliable eigenvalues at higher modes as well as at lower modes, it requires an excessive time to select the primary DOF. The present ELSM shows a slightly larger relative error up to the 25th mode than SEM, and at the higher modes, the relative error oscillates and diverges in the proposed method. However, the error range is only around 3%, and the error value can be tolerated in the structure analysis. In addition, the behavior of the structure is dominated by lower modes. About 25 lower modes are sufficient to represent the behavior of the structure. In Table 3, the computation times of SEM, ELSM, and the full-system method are compared. Although SEM takes less time to solve a reduced system, it takes more time to construct a reduced

Table 1 Time comparison for calculating the reduced system in ELSM

Selection number of elements	Ritz vector calculation, s	Element selection, s	Eigenanalysis, s	Total time, s
20 (9%)[a]		5.29	8.31	54.6
30 (14%)		5.62	12.71	59.3
40 (18%)	41 (120 Ritz vectors constructed)	5.61	16.37	62.98
50 (23%)		5.62	22.96	69.58
60 (28%)		5.78	28.68	75.46

[a]Percentage of total nodes to selected nodes.

Table 2 Time cost of SEM in each number of DOF

Step	DOF	Calculation time, s
Sequential iteration	20	2200
Eigenanalysis	20	1.89
Sequential iteration	30	2184
Eigenanalysis	30	2.1
Sequential iteration	50	1860
Eigenanalysis	50	3.5
Sequential iteration	444	1335
Eigenanalysis	444	20.25

Table 3 Time comparison between SEM, ELSM, and full-system calculation

Selection method	Ritz vector calculation, s	Element selection, s	Sequential iteration, s	Eigen-analysis, s	Total time, s
SEM[a]	None	None	1335	20.25	1355.25
ELSM[a]	41(120 Ritz vectors constructed)	5.61	None	16.37	62.98
Full-system calculation	None	None	None	None	196

[a]Reduced system with 444 DOF.

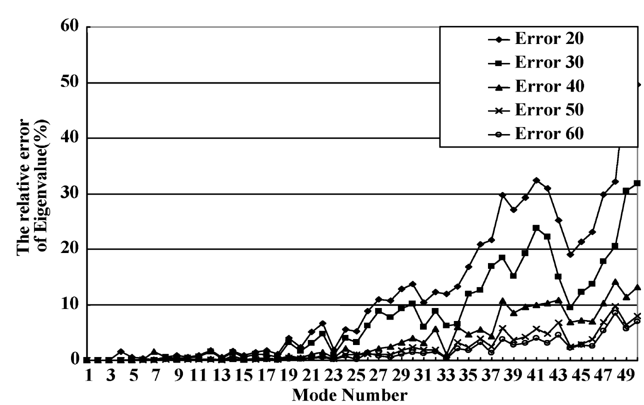
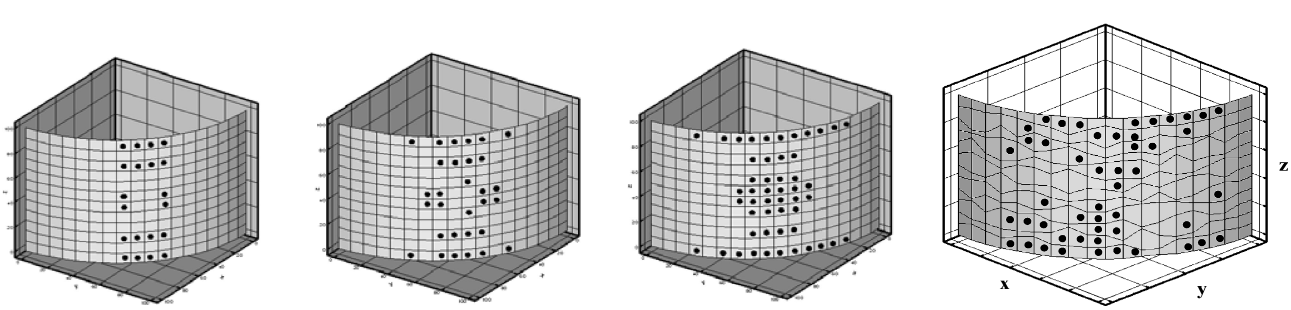


Fig. 4 Relative errors of eigenvalues in each number of selected elements.



a) 20 elements **b) 30 elements** **c) 50 elements** **d) 40 elements, distorted mesh**

Fig. 3 Selected elements (black dots) of cylindrical panel by exact eigenvector of full system and in the distorted mesh.

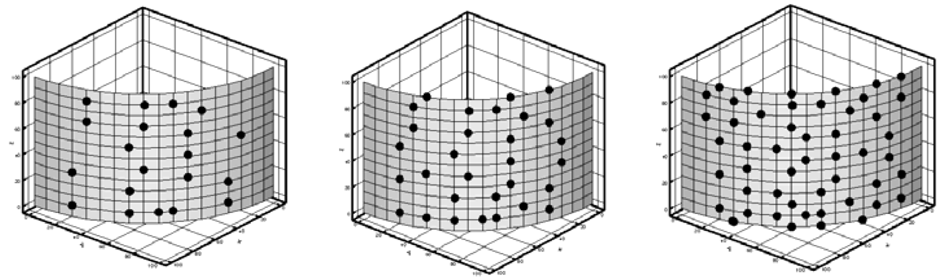


Fig. 5 Selection result by SEM; nodes associated with primary DOF.

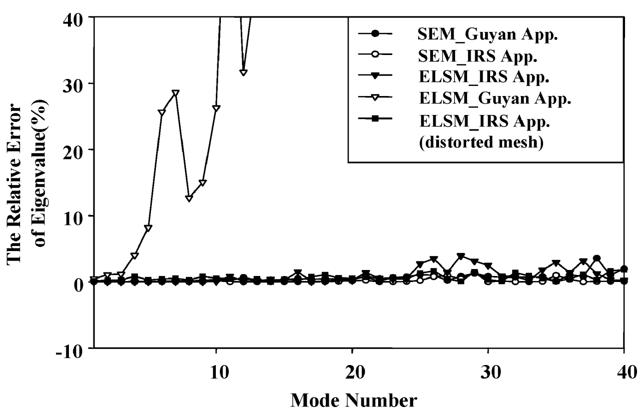


Fig. 6 Relative error comparison of cylindrical panel between ELSM and SEM; both systems have 444 DOF.

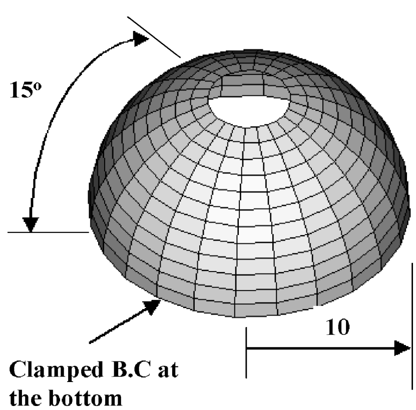


Fig. 7 Configuration and analysis condition of the hemisphere: $E = 6.825 \times 10^7$ Pa, density = 0.01 kg/m^3, Poisson ratio = 0.3, thickness = 0.1 mm, 288 elements, and 325 nodes.

system, and requires more lapse time than that for solving a full system.

B. Hemisphere

A hemisphere model is shown in Fig. 7. The hemisphere is constrained at the bottom by the clamped boundary condition, and it contains a total of 288 elements. To calculate the Rayleigh quotient of each element, 120 Ritz vectors are obtained. With use of these Ritz vectors, the selection locations of 20, 30, 40, and 50 elements are shown in Fig. 8. The Ritz vector procedure takes only 159 s. In case c, (Fig. 8c) the selection procedure takes 8.5 s, and the solving time of the reduced matrix is 10.5 s. From Table 4, the total analysis time for the five different cases is not much different and is around 180 s. Figure 9 shows the selected elements by the eigenvector of the full system. Although the selection locations of the present method do not exactly coincide with the selection location given in Fig. 9, the selection locations determined by the present method are quite close to those given by the exact eigenvector of the full system. In Fig. 10, in the case of 20 selected elements, the relative error is more

Table 4 Time comparison for calculating the reduced system in ELSM

Selection number of elements	Ritz vector calculation, s	Element selection, s	Eigen-analysis, s	Total time, s
20 (14%)[a]		8.5	7.5	175
30 (18%)		8.5	8	175.5
40 (23%)	159 (120 Ritz vectors constructed)	8.5	10.5	178
50 (26%)		8.5	11	178.5
60 (29%)		8.5	16	183.5

[a]Percentage of total nodes to selected nodes.

Table 5 Time cost of SEM in each number of DOF

Step	DOF	Calculation time, s
Sequential iteration	30	5979
Eigenanalysis	30	2
Sequential iteration	50	5860
Eigenanalysis	50	3.5
Sequential iteration	240	2600
Eigenanalysis	240	12.25

Table 6 Time comparison between ELSM and Full system calculation; matrix constructed by ELSM

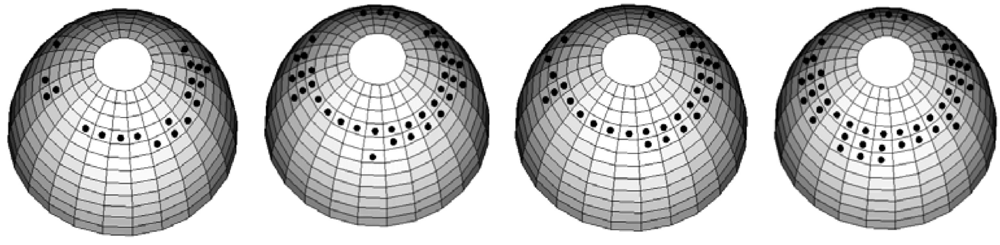
Selection method	Ritz vector calculation, s	Element selection, s	Eigen-analysis, s	Total time, s
ELSM[a] (20%)[b]	450 (100 Ritz vectors constructed)	22	210	682
Full system cal.	None	None	None	1895

[a]Matrix has 738 DOF. [b] Percentage of total nodes to selected nodes.

than 5% in the ranges above the 20th mode. The greater the number of elements selected, the more reliable are the results obtained. The 18% nodes selection (30 elements) shows just around 1% error up to the 30th mode and the 29% node selection (60 elements) almost the exact results. Figure 11 is the configuration with 30 and 50 DOF has selected by SEM. Table 5 is the time comparison according to the selection number of DOF by SEM. In ELSM, the reduced system has 240 DOF. SEM requires 10 times more computing time than ELSM. In Fig. 12, SEM gives reliable results. Although ELSM can not present better results than SEM, the relative error is only around 1∼3 up to the 25th mode; moreover, it has remarkable time efficiency.

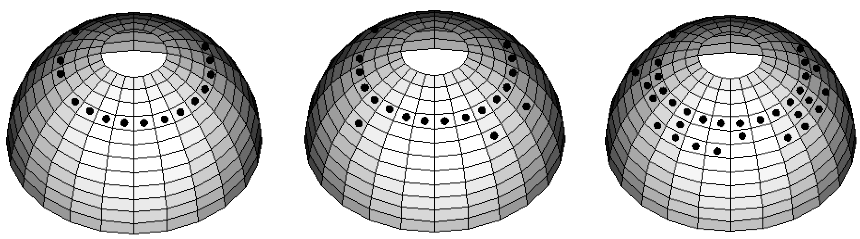
C. Shaft

The geometry in Fig. 13 was obtained using NETGEN.[15] The element type is a tetrahedron with 4 nodes. It consists of 3705 elements and 1193 nodes. Both sides are clamped. For the primary element selection, 100 Ritz vectors are obtained, and it takes 450 s. For the 300-element selection and eigenvalues calculation of the reduced system, Ritz vectors can be obtained in 350 s. The total time is 682 s, which is just one-third of the time required for the full system. The times are given in Table 6. Figure 14 shows the selection results of 300 elements. Only the selected elements on the surface are marked.



a) 20 elements b) 30 elements c) 40 elements d) 50 elements

Fig. 8 Selected elements (black dots) of hemisphere by ELSM.



a) 20 elements b) 30 elements c) 50 elements

Fig. 9 Selected elements (black dots) of hemisphere by exact eigenvector of full system.

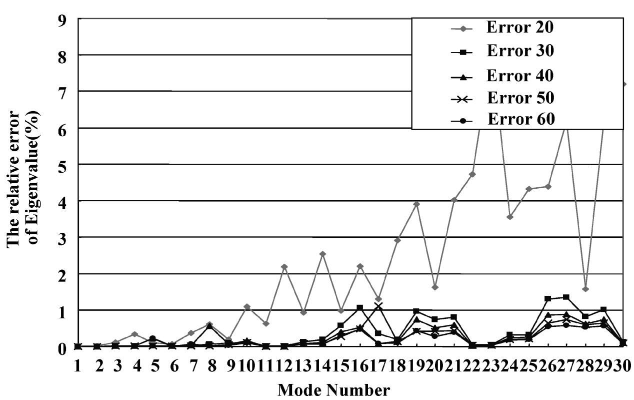


Fig. 10 Relative errors of eigenvalues in each number of selected elements.

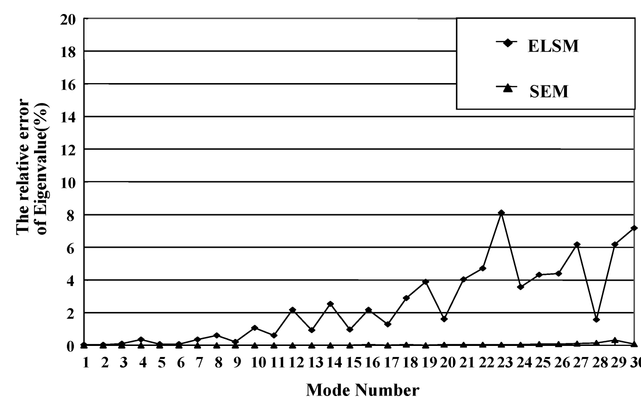
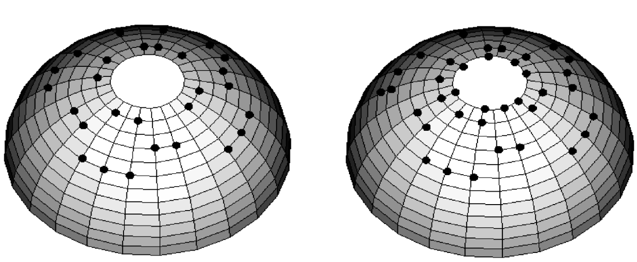


Fig. 12 Relative error comparison of hemisphere between ELSM and SEM; both systems have 240 DOF.



a) 30 DOF b) 50 DOF

Fig. 11 Selection result of SEM; nodes associated with primary DOF.

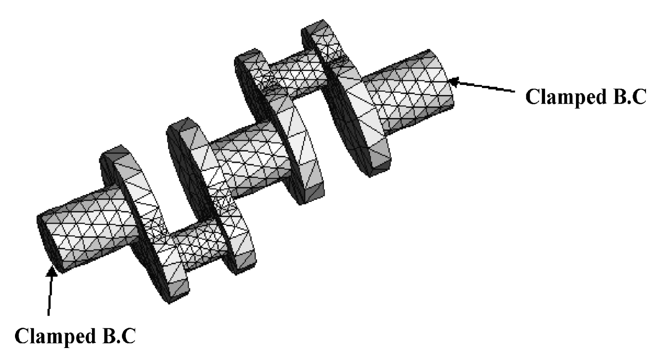


Fig. 13 Boundary condition and material property of shaft: $E = 210$ GPa, mass density = 7850 kg/m^3, 3705 elements (tetrahedron), and 1193 nodes.

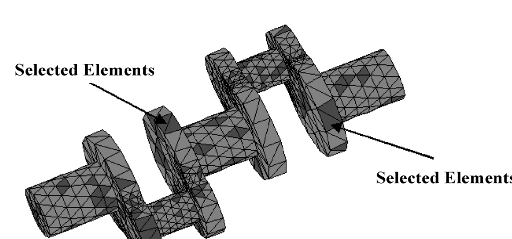


Fig. 14 Selection result of shaft by ELSM.

In Fig. 15, the relative error of the eigenvalues according to selected element number is presented. Both reduced systems with 200/300 selected elements give reliable results. The relative error is just below 1% up to the 30th mode. The case with 300 elements shows only 1~2% relative errors in the first 50 modes. The reduction case with 300 selected elements consists of 738 DOF. The primary DOF are just 20% of those of the global system.

D. Time Comparison

Figure 16 shows computation time comparisons for the example problems in the present study. In each case, the reduced system by ELSM and SEM has the same matrix sizes. Cylindrical panel,

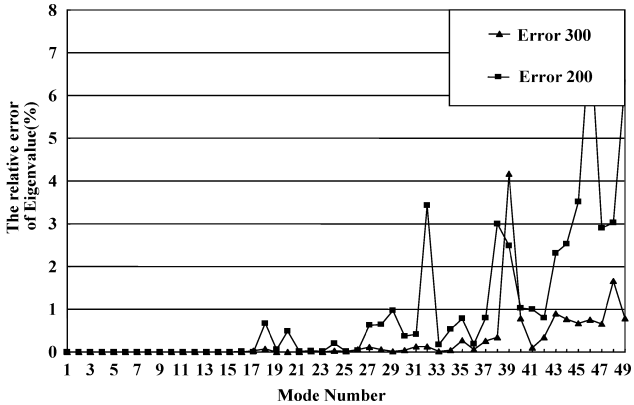


Fig. 15 Relative errors of eigenvalues in each number of selected elements.

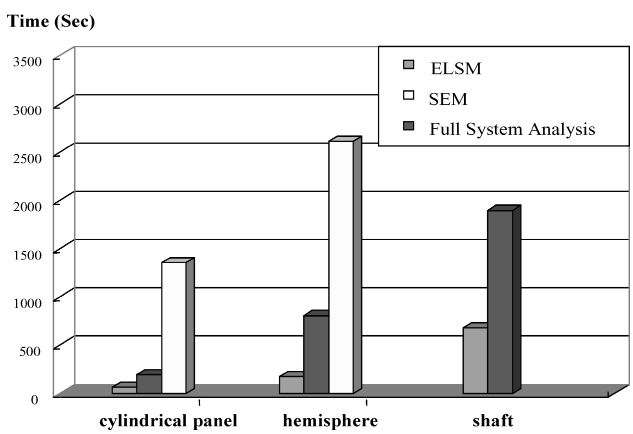


Fig. 16 Time comparison of numerical examples for ELSM, SEM, and full-system analysis: cylindrical panel 444 DOF, hemisphere 240 DOF, and shaft 738 DOF.

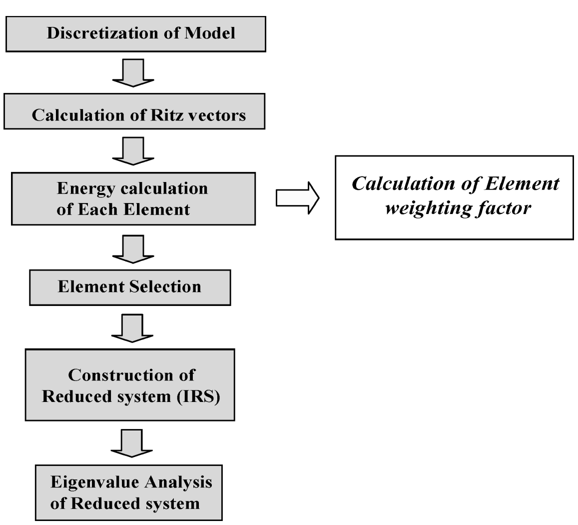


Fig. 17 Algorithm of eigenvalue analysis by constructing a reduced system.

hemisphere, and shaft consist of 444, 240, and 738 DOF, respectively. Although the selection procedure requires additional computing time, the present reduction scheme effectively reduces the computing time compared to the full-model analysis. The present method is three to five times as fast as the full-model analysis. Compared to that of the SEM, the ELSM is 10 times faster in analysis. Although the procedure that computes the Ritz vector in the present method takes much time, the present method is more effective in structural optimization than the traditional method based on repeated analysis. Because of the inefficiency of SEM, the computing time of the eigenvalue problems for a shaft by SEM is substantially larger than that by the full-system analysis. Thus the required computing time of the SEM for this problem is not shown in Fig. 16. ELSM requires only one-third the time needed for the full-system analysis. Figure 17 shows the algorithm of the eigenvalue analysis presented.

V. Conclusions

We proposed a new element-based scheme for constructing a reduced system from a given full system. The reduced system saves a large amount of computing time in the eigenvalue analysis. Previously reported methods were neither accurate nor efficient in the reduction of the number of DOF. In particular, a reliable SEM of only a single DOF per each iteration routine is very slow and inefficient.

To select the primary nodes, we proposed a new method of energy estimation at the element level. The proposed method is independent of the element type. In the finite element analysis, the examples presented the selection of the primary nodes for four-node flat shell elements and four-node tetrahedron solid elements. The algorithm of the eigenvalue analysis in the reduced system is presented. Through the numerical examples, it was demonstrated that the proposed method saves computing time effectively and gives reliable results. In general, SEM gives reliable result at the higher modes, as well as lower modes above the 30th mode, but it takes much time to construct a reduced system. The reduced system of a cylindrical panel with 444 DOF takes about 1300 s, the hemisphere about 2600 s. These times are 10 times the time required by ELSM. Moreover, SEM requires much more time than full-system analysis does. Several numerical examples showed that ELSM does not guarantee higher accuracy than SEM, but it does assure the accuracy in lower modes up to the 20th mode. Roughly, it is constructed with 10% of the total DOF of the global system.

A sufficient number of elements selected increase the eigenvalue accuracy of the global system. From examples, about 20% selections ensure nearly the same results as that of the full-system analysis and gives accurate results at modes above the 30th mode. When the number of the selected elements increases, the reduced matrix size increases. However, it is not always desirable to have a smaller reduced system because the time to calculate the inverse matrix of the slave DOF increases as the size of the final reduced system decreases. Thus, a smaller reduced system does not guarantee faster computation than the larger-sized system. The optimally sized system needs to be found by the present ELSM in a future study.

Finally, this paper presents the scheme for constructing a reduced system by ELSM. It improves the efficiency of SEM. It assures the accuracy of lower modes up to the 20th mode with about 10% DOF of the global system. The analysis of the reduced system with the 20% DOF can also give the reliable results at the modes above the 30th mode.

In the design problem of a dynamic system, we need to compute eigenvalue and eigenvector sensitivities. Large-scaled problems require a considerable amount of computing time to obtain design sensitivities. Thus, it is desirable to get the sensitivities within the reduced system analysis for the computational efficiency. Sensitivity analysis of the reduced system is now in progress.

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